AFOSR-TN-59-1102

CALIFORNIA INSTITUTE OF TECHNOLOGY

Antenna Laboratory

Technical Report No. 21

A NOTE ON CURRENTS ON A QUADRATIC SURFACE

Georges G. Weill

This research was supported by the U.S. Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract Number AF18(600)-1113.

A NOTE ON CURRENTS ON A QUADRATIC SURFACE

by

Georges G. Weill

ABSTRACT

Although no explicit general solution is known for the vector integral equation satisfied by the current density vector on a conducting surface, it is shown in the following report that the vector equation can be "scalarized" in the case of a quadratic conducting surface.

A NOTE ON CURRENTS ON A QUADRATIC SURFACE

1. Introduction

It has been shown by Fock (Ref. 1) that the surface current density vector satisfies the following integral equation:

$$\vec{j}(r) = 2\vec{j}_o(r) + \frac{1}{2\pi}\int_S (\vec{n} \times \vec{j}(r') \times (\vec{r} - \vec{r}\,') \, g(R))ds'$$

when $g(R) = \dfrac{(1 - ikR)e^{ikR}}{R^3}$.

In this equation, R is the length of the chord joining the two points on the surface S defined by $\vec{r}$ and $\vec{r}\,'$

$\vec{n}$ is the unit normal vector at $\vec{r}$

ds' the surface element at $\vec{r}\,'$

k the absolute value of the wave vector

The quantity $\vec{j}_o$ is an "external" current density defined by

$$\vec{j}_o(r) = \frac{c}{4\pi}\,[\vec{n} \times \vec{H}_o]$$

where $\vec{H}_o$ is the magnetic field of the incident wave on the surface (external field).

In the general case, by projection on a convenient system of coordinates (ξ,η) on the surface, one gets:

$$j_\eta = 2j_{\eta_o} + \mathcal{L}_1\, j'_\eta + \mathcal{L}_2\, j'_\xi$$

$$j_\xi = 2j_{\xi o} + \mathcal{M}_1\, j'_\xi + \mathcal{M}_2\, j'_\eta$$

where the $\mathcal{L}_i$ and $\mathcal{M}_i$ $(i = 1,2)$ are linear integral operators.

We shall see now under which conditions the preceding integral equations simplify. In particular, one could ask the following question: Can we have

$$\mathcal{L}_2 = \mathcal{M}_2 = 0 \quad , \quad \text{or} \qquad \mathcal{L}_1 = \mathcal{M}_1 = 0 \quad ?$$

2. Characterization of the Surface S

Let us take a system (ξ, η) of coordinate curve or S , not necessarily orthogonal. At each point we define the tangent to the ξ line by $\vec{a}_\xi$, its unit vector. $\vec{a}_\eta$ is the unit vector of the tangent to the η-line. We have

$$\vec{j}(r') = j'_\xi \, \vec{a}'_\xi + j'_\eta \, \vec{a}'_\eta$$

$$\vec{n}(r) = \frac{\vec{a}_\xi \times \vec{a}_\eta}{\left| \vec{a}_\xi \times \vec{a}_\eta \right|} \quad .$$

Therefore:

$$\vec{n}(r) \times \left[\vec{j}(r') \times (\vec{r} - \vec{r}') \right] = \frac{\left[\vec{a}_\xi \times \vec{a}_\eta \right]}{\vec{a}_\xi \times \vec{a}_\eta} \left[j(r') \times (\vec{r} - \vec{r}') \right]$$

$$= \frac{(\vec{j}(r'), (\vec{r} - \vec{r}'), \vec{a}_\xi)}{\vec{a}_\xi \times \vec{a}_\eta} \, \vec{a}_\eta - \frac{(\vec{j}(r'), (\vec{r} - \vec{r}'), \vec{a}_\eta)}{\vec{a}_\xi \times \vec{a}_\eta} \, \vec{a}_\xi$$

where $(\vec{a}, \vec{b}, \vec{c})$ is the mixed product $(\vec{a} \cdot \vec{b} \times \vec{c})$.

We have furthermore:

$$(\vec{j}(r'), (\vec{r} - \vec{r}'), \vec{a}_\xi) = (j'_\xi \vec{a}'_\xi + j'_\eta \vec{a}'_\eta, (\vec{r} - \vec{r}'), \vec{a}_\xi)$$

$$(\vec{j}(r'), (\vec{r} - \vec{r}'), \vec{a}_\eta) = (j'_\xi a'_\xi + j'_\eta \vec{a}'_\eta, (\vec{r} - \vec{r}'), \vec{a}_\eta)$$

The coefficient of $\vec{a}_\eta$ is independent of j'_ξ if, and only if

$$(\vec{a}'_\xi, (\vec{r} - \vec{r}'), \vec{a}_\xi) = 0 \qquad (1)$$

and the coefficient of $\vec{a}_\xi$ is independent of j'_η if, and only if,

$$(\vec{a}'_\eta, (\vec{r} - \vec{r}'), \vec{a}_\eta) = 0 \qquad (1')$$

for all couple of points $\vec{r}$ and $\vec{r}'$. In the same way, the coefficient of a_η is independent of j'_η if

$$(\vec{a}'_\eta, (\vec{r} - \vec{r}'), \vec{a}_\xi) = 0 \qquad (2)$$

and the coefficient of $\vec{a}_\xi$ is independent of j'_ξ if

$$(\vec{a}'_\xi, (\vec{r} - \vec{r}')\vec{a}_\eta) = 0 . \qquad (2')$$

Conditions (1) and (1') mean that a_ξ, a'_ξ, $(\vec{r} - \vec{r}')$ are in the same plane, or $\vec{a}'_\xi$ always meets $\vec{a}_\xi$, and in the same way $\vec{a}_\xi$ always meets $\vec{a}_\eta$.

It is almost obvious that S in this case is a plane or a pencil of planes. Take two $\vec{a}_\xi$, say $\vec{a}_{\xi 1}$ and $\vec{a}_{\xi 2}$, which interact at a point A (if they exist). Every $\vec{a}_\xi$ has to intersect $\vec{a}_{\xi 1}$ and $\vec{a}_{\xi 2}$, hence has to be either in the plane defined by $\vec{a}_{\xi 1}$ and $\vec{a}_{\xi 2}$, or go through A. In the first case, all $\vec{a}_\xi$ lie in a plane, and so do points of contact. Hence S is a plane. In the second case, all $\vec{a}_\xi$

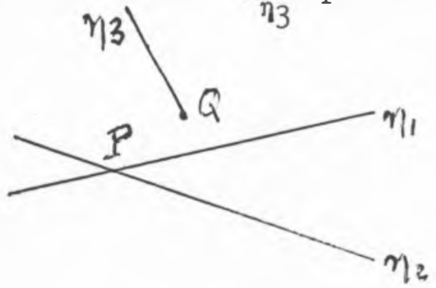
go through some point A . If in this second case all $\vec{a}_\eta$ are in a plane we are through. Otherwise, all $\vec{a}_\eta$ intersect at some point B . If A and B are distinct, all planes tangent to S go through the line AB.

Take AB for z axis. If $f(x,y,z) = 0$ is the equation of S , we have $f'_z = 0$, hence S is a cylinder. Take the cross section by $z = 0$; a curve whose tangents all go through O , hence is a family of straight lines through O . S is then a family of planes intersecting along Oz . A and B must be different, otherwise $\vec{a}_\xi$ and $\vec{a}_\eta$ would not be defined by coordinate lines.

To sum up S is a plane or a pencil of planes. This covers for the case of fixed limits in the integral; plane, parallel plates, wedges, corners.

Conditions 2 and 2' mean that every $\vec{a}_\eta$ intersects every $\vec{a}_\xi$ and every $\vec{a}_\xi$ intersects every $\vec{a}_\eta$. If, for example, three $\vec{a}_\eta$ exist, two of which are not coplanar, all $\vec{a}_\xi$ intersecting those three $\vec{a}_\eta$, the $\vec{a}_\xi$ are a family of generators of quadratic surface, which coincides then with S . The $\vec{a}_\eta$ are then the other family.

Suppose now $\vec{a}_{\eta 1}$ and $\vec{a}_{\eta 2}$ are coplanar, and let P be their intersection. $\vec{a}_{\eta 3}$ pierces their plane at Q ; all $\vec{a}_\xi$ are either in the plane and go through Q , or in the plane (P,η_3) and go through P . We have then, two planes for S . If $\vec{a}_{\eta 1}$ $\vec{a}_{\eta 2}$ $\vec{a}_{\eta 3}$ are always coplanar, then S is their plane.



To sum up, we have proved the following:

Proposition A. The vector equation for the currents on a conducting surface S can be "scalarized" if S is a non-degenerate quadratic surface.

3. The Degenerate Quadratic Case

We just proved that in the case of a pencil of planes the previous proposition was true. In the case of a cone (or cylinder) our two coordinate curve systems become one coordinate curve system. Let it be the ξ system.

Then,

$$\vec{n}(\vec{r}) \times \left[j(\vec{r}\,')\times(\vec{r} - \vec{r}\,') \right] = (j_\eta' \vec{a}_\eta' , \vec{r}-\vec{r}\,', \vec{a}_\xi)\vec{a}_\eta$$

$$- (j_\xi' \, a_\xi' + j_\eta' \, \vec{a}_\eta', \vec{r}-\vec{r}\,', \vec{a}_\eta)\vec{a}_\xi \; .$$

Take any η system, for instance the orthogonal trajectories of the $\vec{a}_\xi$ for the cone (or cylinder). Then, the equation for $\vec{j}_\eta$ becomes independent of $\vec{j}_\xi'$. Once this equation is solved by a simple substitution in the equation for $\vec{j}_\xi$ one gets an equation involving only $\vec{j}_\xi$, and we have to solve again two scalar equations.

We get then,

Proposition B. The vector equation for the currents on a conducting surface can be "scalarized" if S is a quadratic surface.

It is noteworthy to remember that the wave equation is separable only in coordinate systems where the coordinate surfaces are quadratic surfaces. It might be interesting to investigate the connection between separability and scalarization.

4. Some More Remarks

One might write down an explicit equation for the kernel $K = (\vec{a}'_\eta, \vec{r}-\vec{r}', \vec{a}_\eta)f(R)$ in the non-degenerate case, using Plueckerian coordinates $(X_{\eta'},Y_{\eta'},Z_{\eta'},L_{\eta'},M_{\eta'},N_{\eta'})$ for η', $(X_\eta,Y_\eta,Z_\eta,L_\eta,M_\eta,N_\eta)$, the one for η, [(X,Y,Z) being a unit vector]

$$K = (L_\eta X_{\eta'} + M_\eta Y_{\eta'} + N_\eta Z_{\eta'} + L_{\eta'} X_\eta + M_{\eta'} Y_\eta + N_{\eta'} Z_\eta)f(R)$$

$$\frac{dS'}{|\vec{a}_\xi \times \vec{a}_\eta|} = \frac{|\vec{a}_\xi{}' \times \vec{a}'_\eta|}{|\vec{a}_\xi \times \vec{a}_\eta|} \, d'_\xi \, d'_\eta \quad .$$

Expressions like $|\vec{a}_\xi{}' \times \vec{a}_\eta{}'|$ can be absorbed by $\vec{j}'$, $|\vec{a}_\xi \times \vec{a}_\eta|$ by $\vec{j}$ so they do not appear explicitly.

After such an absorption, the kernel becomes a symmetric one with respect to the primed and unprimed letters, which was obvious on the vector expression. The interesting thing is that the bracket is of "dyadic" type.

Although the bracket is symmetric, K is not Hermitian:

$$\overline{K(r,r')} \neq K(r',r) \qquad \text{and} \qquad K(r,r') = K(r',r)$$

Some advances in the theory of non-Hermitian symmetric kernels (in particular the study of completeness) would enable us to derive some topological results on the current lines. It is hoped that some treatment will be given in a subsequent report.

5. Conclusion

The preceding results have reduced the vector equation for the quadratic case to two scalar independent integral equations. However, even in this case a closed form solution is not known for the corresponding integral equation.

It can be easily checked that when one assimilates the surface to its tangent plane, and assuming that the solution is of local character, the equation can be solved by the convolution theorem. (See, for instance, the method in 2.)

BIBLIOGRAPHY

1. V. Fock, "The Distribution of Currents Induced by a Plane Wave on the Surface of a Conductor", Jour. Phys. USSR 10, 130-136 (1948).

2. J. A. Cullen, "Surface Currents Induced by Short Wavelength Radiation", Phys. Rev. 109, No.6, 1863 (1958).

DISTRIBUTION OF REPORTS

AF18(600)-1113

Commander, ATTN: SRY
AF Office of Scientific Res. 3
Washington 25, D. C.

Commander, ATTN: SRLT
AF Office of Scientific Res. 2
Washington 25, D. C.

Commander, ATTN: WCOSI
Wright Air Development Center 4
Wright-Patterson A.F. Base
Ohio

Commander, ATTN: CROT
AF Cambridge Research Center 1
L.G. Hanscom Field
Bedford, Massachusetts

Commander, ATTN: RCSSLD
Rome Air Development Center 1
Griffiss A.F. Base
Rome, New York

Commander, European Office 2
Air Research and Dev. Command
The Shell Building
47 rue Cantersteen
Brussels, Belgium

P.O. Box AA 1
Wright-Patterson A.F. Base
Ohio

Armed Services Tech.Inf.Agency 10
ATTN: TIPDR
Arlington Hall Station
Arlington 12, Virginia

Director of Res. and Dev. 1
Headquarters, USAF
ATTN: AFDRD
Washington 25, D. C.

Office of Naval Research 1
Department of the Navy
ATTN: Code 420
Washington 25, D. C.

Director, Naval Research Lab. 1
ATTN: Technical Inf. Officer
Washington 25, D. C.

Director, Army Research Office 1
Research and Development
Department of the Army
Washington 25, D. C.

Chief, Physics Branch 1
Division of Research
U.S. Atomic Energy Commission
Washington 25, D. C.

U.S. Atomic Energy Commission 1
Tech. Information Extension
P.O. Box 62
Oak Ridge, Tennessee

Natl. Bureau of Standards 1
Library, Room 203, NW Bldg.
Washington 25, D. C.

Physics Program 1
National Science Foundation
Washington 25, D. C.

Director, Office of Ordnance Res. 1
Box CM, Duke Station
Durham, North Carolina

Director, Dept. of Commerce 1
Office of Technical Services
Washington 25, D. C.

ARO, Inc. 1
ATTN: AEOI
Tullahoma, Tennessee

Commander, Air Proving Ground 1
ATTN: ACOT
Eglin A.F. Base, Florida

Commander 1
A.F. Flight Test Center
ATTN: FTOTL
Edwards A.F. Base, California

Commander 1
ATTN: ORDXR-OTL
Army Rocket and Guided Missile Ag.
Redstone Arsenal, Alabama

Commander 1
AF Special Weapons Center
ATTN: SWOI
Kirtland A.F. Base, New Mexico

Commander 1
AF Missile Development Center
ATTN: HDOI
Holloman A.F. Base, New Mexico

Commandant, ATTN: MCLI 1
A.F. Institute of Technology
Wright-Patterson A.F.Base, Ohio

Commander, ATTN: WDSOT 1
AF Ballistic Missile Division
AF Unit Post Office
Los Angeles 45, California

Commander 4
Air Res. and Dev. Command
Andrews AF Base,
Washington 25, D. C.
ATTN: RDR 1 copy
RDRR 1 copy
RDSFI-1 2 cc

Commanding General 1
U.S. Army Signal Corps
Research and Development Lab.
ATTN: SIGFM/EL-RPO
Ft. Monmouth, New Jersey

National Aeronautics and Space Administration 6
Washington 25, D. C.

Advanced Research Projects Ag. 1
Washington 25, D. C.

Rand Corporation 1
1701 Main Street
Santa Monica, California

Director, AUL 9663 1
Air University Library
Maxwell A.F. Base, Alabama

Chairman (DRB/DSIS) 1
Canadian Joint Staff
2450 Massachusetts Ave, NW
Washington 25, D. C.

Applied Mechanics Review 1
Southwest Research Institute
8500 Culebra Road
San Antonio 6, Texas

Inst. of the Aeronautical Sc. 1
2 East 64th Street
New York 16, New York
ATTN: Librarian

Exchange and Gift Division 1
Library of Congress
Washington 25, D. C.

Remington Rand Univac 1
Div. of Sperry Rand Corp.
19th St. and W.Allegheny Ave.
Philadelphia 29, Pennsylvania

Director, 1
Res. Lab. for Electronics
Massachusetts Inst. of Tech.
Cambridge 39, Massachusetts

Dean F. E. Terman 1
Electronics Research Laboratory
Stanford University
Stanford, California

Professor E. Weber 1
Microwave Research Laboratory
Polytechnic Inst.of Brooklyn
Brooklyn, New York

Head 1
G.E. Microwave Laboratory
601 California Avenue
Palo Alto, California

Professor T. E. Tice 1
Dept. of Electrical Engineering
Ohio State University
Columbus 10, Ohio

Dr. R. W. P. King 1
Harvard University
Cambridge, Massachusetts

General Electric Company 1
Missile and Ordnance Systems
3198 Chestnut Street
Philadelphia 4, Pennsylvania
ATTN: Library Manager

Professor John Hoffman 1
Dept. of Electrical Engineering
University of Illinois
Champaign, Illinois

Professor W. G. Shepherd 1
Dept. of Electrical Engineering
University of Minnesota
Minneapolis, Minnesota

Technical Library 1
Research and Dev. Labs.
Hughes Aircraft Company
Culver City, California

Dr. R. E. Hutter, Chief Eng. 1
Sylvania Elec. Products Inc.
P. O. Box 997
Mountain View, California

Professor P. Kusch 1
Radiation Laboratory
Columbia University
New York, New York

Dr. Donald E. Kerr 1
John Hopkins University
Department of Physics
Baltimore 18, Maryland

Dr. D. W. Healy
Electrical Engineering Dept. 1
Syracuse University Res. Inst.
Syracuse 10, New York

U. of Southern California 1
Electrical Engineering Dept.
Los Angeles 7, California
Attn: Z. Kaprielian

Stanford University 1
Microwave Research Laboratory
Stanford, California
Attn: M. Chodorow

University of Minnesota 1
Electrical Engineering Dept.
Minneapolis 14, Minnesota
Attn: H. Oskam

Northeastern University 1
Physics Department
Boston, Massachusetts
Attn: G. Lanza

University of New Hampshire 1
Department of Physics
Durham, New Hampshire
Attn: L. Mower

Rutgers University 1
Microwave Laboratory
New Brunswick, New Jersey
Attn: M. Sirkis

New York University 1
Inst. of Mathematical Sciences
25 Waverly Place
New York 3, New York
Attn: M. Kline

Oxford University 1
Oxford, England
Attn: H. Motz

University of Illinois 1
Dept. of Electrical Engineering
Urbana, Illinois
Attn: P. Coleman

Stevens Inst. of Technology 1
Physics Department
Hoboken, New Jersey
Attn: W. Bostick

University of California 1
Electronics Research Laboratory
Berkeley 4, California
Attn: J. Whinnery

Office of Naval Research 1
346 Broadway
New York 12, New York
Attn: I. Rowe

National Aeronautics and Space
Administration 1
1520 H Street, N.W.
Washington 25, D. C.
Attn: B. Mulcahy

Brooklyn Polytechnic Institute 1
Microwave Research Institute
55 Johnson Street
Brooklyn 1, New York
Attn: N. Marcuvitz

Dr. Glenn H. Keitel 1
Advanced Programs Section
Philco Corporation
3875 Fabian Way
Palo Alto, California